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HARMONIC ANALYSIS WITH RESPECT TO ALMOST
INVARIANT MEASURES: A FIRST APPROXIMATION

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1. INTRODUCTION. One of the central concerns of harmonic analysis is the study of the commutative Banach algebra $L_1(G)$ consisting of all the equivalence classes of complex valued functions on the locally compact abelian (LCA) group G which are integrable with respect to Haar measure m on G and with the usual convolution product.

$$f * g(s) = \int_G f(st^{-1})g(t)dm(t), \quad f, g \in L_1(G).$$

Since m is invariant under translation by elements of G it is trivially apparent that $\{T_s m | s \in G\}$ spans a one dimensional space of measures, where $T_s m(E) = m(Es)$. In light of this observation it seems natural to attempt an investigation of algebras which occur in a manner analogous to the group algebra $L_1(G)$ but where Haar measure is replaced by a more general type of measure, in particular by an almost invariant measure; that is, by a regular complex valued Borel measure μ on G such that $\{T_s \mu | s \in G\}$ spans a finite dimensional space of measures.

In the sequel we shall construct such algebras and examine various aspects of their structure.

Before proceeding to the construction of these algebras we wish to recall some basic facts about almost invariant measures. $FDT(G)$ will denote the space of all continuous complex valued functions h on a LCA group G such that $\{T_s h | s \in G\}$ spans a finite dimensional space of functions and where $T_s h(t) = h(ts)$, that is, $FDT(G)$ is the space of continuous almost invariant functions on G .

A characterization of almost invariant measures is given by the following theorem.

THEOREM 1. Let G be a LCA group. Then the following are equivalent:

- i) μ is an almost invariant measure on G .
- ii) There exists a unique $h \in \text{FDT}(G)$ such that $d\mu = h \, dm$.

The proof of this and other results related to almost invariant measures can be found in [4,5,6]. In the next paragraphs we shall restrict our attention to $h \in \text{FDT}(G)$ for which the supremum norm h_∞ is finite. In this case h is a trigonometric polynomial.

2. THE SPACE $L_1(h)$. In the beginning of our development we shall have to make certain terminological distinctions which ultimately will disappear. If μ is any regular complex valued Borel measure on a LCA group G then we shall denote by $L_1(\mu)$ the Banach space of all (equivalence classes) of functions on G which are integrable with respect to μ and with the norm

$$\|f\|_\mu = \int_G |f(t)| d|\mu|(t), \quad f \in L_1(\mu).$$

We shall reserve the notation $L_1(G)$ for the Banach algebra arising from $L_1(m)$ by the introduction of convolution multiplication.

The linear space underlying the group algebra $L_1(G)$ is of course

$$\begin{aligned} L_1(m) &= \{f \mid \int_G |f(t)| dm(t) < \infty\} \\ &= \{f \mid \int_G |T_{s^{-1}} f(t)| dm(t) < \infty, s \in G\}. \end{aligned}$$

Having noted this, the following definition defines a natural replacement for $L_1(m)$ when we replace m by an almost invariant measure μ .

DEFINITION 1. Let μ be an almost invariant measure on a LCA group G such that $d\mu = h \, dm$ and $h_\infty < \infty$. Then

$$L_1(h) = \{f \mid f \text{ is Borel measurable, } \int_G |T_{s^{-1}} f(t)| d|\mu|(t) < \infty, s \in G\}.$$

Clearly $L_1(h)$ is a linear subspace of $L_1(\mu)$, and a

relatively straightforward argument involving the almost periodicity of h establishes the following result.

THEOREM 2. Let G be a LCA group, $h \neq 0$ a trigonometric polynomial on G . Then the following are equivalent:

i) $f \in L_1(m)$.

ii) $f \in \mathcal{L}_1(h)$.

REMARKS. a) In the definition of $\mathcal{L}_1(h)$ it is not sufficient for our purposes to require only that $\int_G |f(t)| d|\mu|(t) < \infty$, since we clearly wish to have $\mathcal{L}_1(h)$ closed under translation by G and it is easy to construct μ , f and s such that

$$\int_G |f(t)| d|\mu|(t) < \infty \text{ but } \int_G |T_{s^{-1}} f(t)| d|\mu|(t) = \infty. \text{ For example}$$

let $G = \mathbb{R}$, the additive group of the real line, $d\mu(t) = \sin t \, dm(t)$ and $f = \chi_{[-\pi/2, \pi/2]}$ where

$$g(t) = \begin{cases} 1/\sin t, & \sin t \neq 0 \\ 0 & , \sin t = 0. \end{cases}$$

Then $f \in L_1(\mu)$ but $T_{-\pi/4} f \notin L_1(\mu)$.

b) Also it is easily seen that the requirement $h_\infty < \infty$ is necessary to insure that $\mathcal{L}_1(h) \neq \{0\}$.

c) In general we shall deal with the function h associated with an almost invariant measure by Theorem 1 rather than the measure itself.

An immediate consequence of Theorem 2 is that $\mathcal{L}_1(h)$ can be considered as a Banach space under the norm $\|f\| = \|f\|_m$, $f \in L_1(m)$. Our next task will be to introduce an appropriate multiplication into $\mathcal{L}_1(h)$ which involves, in an essential fashion, the function h and which reduces to the usual convolution product when $h \equiv 1$.

The best definition seems to be the following one.

DEFINITION 2. Let G be a LCA group, $h \neq 0$ a trigonometric polynomial on G . If $f, g \in \mathcal{L}_1(h)$ then

$$f \circ g(s) = f h * g h(s) = \int_G f h(st^{-1}) g h(t) dm(t).$$

Using Theorem 2 and the previous remark one can readily prove the next theorem.

THEOREM 3. Let G be a LCA group, $h \neq 0$ a trigonometric polynomial on G , $\|h\|_\infty \leq 1$. Then $\mathcal{L}_1(h)$ is a commutative non-associative Banach algebra under the multiplication \circ .

REMARKS. a) By a nonassociative Banach algebra we mean a Banach space equipped with a multiplication which satisfies all the usual conditions to make it a Banach algebra with the possible exception of the associative law of multiplication. Thus the validity of the associative law is not assumed though it may hold in particular cases. If the associative law fails to be valid then the algebra is said to be not associative. This is the standard terminology for such algebras [10].

b) It is not difficult to choose G and h such that $\mathcal{L}_1(h)$ is not associative. For example let G be any LCA group distinct from the identity and let $\chi \in \hat{G}$, the dual group of G , be a continuous character on G which is not identically one. If we set $h = (\cdot, \chi)$ then some routine computations and some well known properties of the Fourier transform show that one can choose $f, g, k \in \mathcal{L}_1(h)$ such that $[(f \circ g) \circ k]^\wedge(e') \neq [f \circ (g \circ k)]^\wedge(e')$, — where e' here denotes the identity element in G . The uniqueness of the Fourier transform $^\wedge$ then shows that $f \circ (g \circ k) \neq (f \circ g) \circ k$, that is, $\mathcal{L}_1(h)$ is not associative.

c) The question of associativity will be examined more thoroughly below (Theorems 12, 13, 14). Obviously if the associative law does hold then $\mathcal{L}_1(h)$ is a commutative Banach algebra in the usual sense.

d) Clearly $\mathcal{L}_1(h) = L_1(G)$ when $h \equiv 1$.

e) The restriction $\|h\|_\infty \leq 1$ is only one of convenience. If $1 < \|h\|_\infty < \infty$ then one defines $f \circ g = f h * g h / (\|h\|_\infty)^2$.

With this multiplication Theorem 3 is again valid.

f) Another candidate for the multiplication in $\mathcal{L}_1(h)$ would be $f \circ g = f * g h$. But this is even less tractable than the product we have chosen since it is not in general a commutative multiplication.

Since the algebras $\mathcal{L}_1(h)$ we wish to study are not always associative we must exercise some care in appealing to standard theorems about Banach algebras during our investigations. For example, the usual proofs of the Gelfand-Mazur theorem, and hence the Gelfand representation theory, depend on the associativity of the algebra. Rather than attempt to isolate those results in the theory of Banach algebras which remain valid if the associative law fails, we shall instead conduct a more direct assault on the problems we wish to study in the algebras $\mathcal{L}_1(h)$. Of course the associative law holds we shall make full use of the machinery of the theory of Banach algebras.

3. MULTIPLICATIVE LINEAR FUNCTIONALS ON $\mathcal{L}_1(h)$. As usual any continuous homomorphism of $\mathcal{L}_1(h)$ onto \mathbb{C} , the field of complex numbers, will be called a multiplicative linear functional on $\mathcal{L}_1(h)$. An argument essentially like the one used to characterize the multiplicative linear functionals on $L_1(G)$ establishes the next theorem.

THEOREM 4. Let G be a LCA group, $h \neq 0$ a trigonometric polynomial on G , $\|h\|_\infty \leq 1$. Then the following are equivalent:

- i) F is a multiplicative linear functional on $\mathcal{L}_1(h)$
- ii) There exists a bounded continuous function α on G such that

$$F(f) = \int_G f(t) \alpha(t) dm(t), \quad f \in \mathcal{L}_1(h),$$

and

$$h(t) h(s) \alpha(ts) = \alpha(t) \alpha(s), \quad t, s \in G.$$

Thus we see that the study of the multiplicative linear functionals on $\mathcal{L}_1(h)$ reduces to an examination of the functional equation

$$h(t) h(s) \alpha(ts) = \alpha(t) \alpha(s), \quad t, s \in G,$$

where $h \neq 0$ is a trigonometric polynomial on G . Obviously $\alpha \equiv 0$ is always a solution, and in many cases it is the only solution. Our next task is to give criteria for when continuous solutions $\alpha \neq 0$ exist and describe such solutions.

As will become quickly apparent our solution to this problem is not overly elegant, but consists rather of a successive whittling down of the problem to a final relatively tractable situation. A number of the results presented in the next section depend on neither the fact that h is a trigonometric polynomial nor on the continuity of the functions involved, but are purely combinatoric in nature. On the other hand the important results for our purposes do depend on the fact that a trigonometric polynomial is a bounded, continuous almost periodic function, and the main theorems will be stated in terms of such functions. By an almost periodic function we shall always mean a bounded continuous almost periodic function.

4. SOLUTION OF $h(t) h(s) \alpha(ts) = \alpha(t) \alpha(s)$.

The case where h is a constant is easily handled and provides us with the following theorem.

THEOREM 5. Let G be a LCA group and $h(t) \equiv h(e)$, $t \in G$. Then the following are equivalent:

- i) $\alpha \neq 0$ is a continuous function on G such that $h(t) h(s) \alpha(ts) = \alpha(t) \alpha(s)$, $t, s \in G$.
- ii) There exists a unique $\psi \in \hat{G}$ such that $\alpha(t) = h(e)^2 (t, \psi)$, $t \in G$.

Of course when h is a constant then $\mathcal{L}_1(h)$ is essentially the same as $L_1(G)$ since $f \circ g = h(e)^2 (f * g)$. In view of the previous theorem we can restrict our attention to nonconstant functions h .

Before we proceed further we must introduce another definition.

DEFINITION 3. Let k be a function on a LCA group G . Then we set

$$z(k) = \{t \mid t \in G, k(t) = 0\},$$

and

$$\tilde{z}(k) = G \cap z(k) = \{t \mid t \in G, k(t) \neq 0\}.$$

The set $\tilde{z}(k)$, for various functions k , will play an important role in the solution of the functional equation. For example $\tilde{z}(\alpha)$, where $\alpha \neq 0$ is a solution of the functional equation, will be a certain type of subgroup contained in $\tilde{z}(h)$, the existence of which will be a necessary and sufficient condition for the existence of a non-trivial solution to the functional equation.

It is perhaps well to note that when k is continuous the set $\tilde{z}(k)$ is not the support of k in the usual sense. The latter set is of course the closure of $\tilde{z}(k)$ in G .

We cannot give all the details of the solution of the functional equation here, but rather will state the main lemmas and theorems and give an indication of the nature of some of the proofs.

First we have a purely combinatoric lemma. e denotes the identity in G .

LEMMA 1. Let G be a LCA group and suppose h, α are functions on G such that $h(t)h(s)\alpha(ts) = \alpha(t)\alpha(s)$, $t, s \in G$. Then:

- i) $\tilde{z}(\alpha) \subset \tilde{z}(h)$.
- ii) $z(h) \subset z(\alpha)$.
- iii) If $t \in \tilde{z}(\alpha)$ then $t^k \in \tilde{z}(\alpha)$, $k = 1, 2, 3, \dots$.
- iv) If $e \in z(\alpha)$ then $t \in \tilde{z}(\alpha)$ implies $t^{-k} \in z(h)$, $k = 1, 2, 3, \dots$.

Using this lemma we obtain the following result.

THEOREM 6. Let G be a LCA group, h a nonconstant almost periodic function on G and suppose $h(t)h(s)\alpha(ts) = \alpha(t)\alpha(s)$, $t, s \in G$. Then the following are equivalent:

- i) $e \in z(\alpha)$.
- ii) $\alpha \equiv 0$.

We shall give some indication of the proof of the theorem since it is characteristic of a type of argument which is used at several stages in the solution of the functional equation.

PROOF. (Sketch) If $e \in z(\alpha)$ we have two cases to consider to which we apply Lemma 1.

1) $e \in \tilde{z}(h)$. Then if $\alpha \neq 0$ there exists some $t \in \tilde{z}(\alpha) \subset \tilde{z}(h)$. But then $h(t) h(e) \alpha(t) = \alpha(t) \alpha(e) = 0$ implies that $e \in z(h)$, contrary to assumption. Thus $\alpha \equiv 0$.

2) $e \in z(h)$. If $\alpha \neq 0$ then there exists $t \in \tilde{z}(\alpha) \subset \tilde{z}(h)$. For such t we also have $t^k \in \tilde{z}(\alpha) \subset \tilde{z}(h)$, $t^{-k} \in z(h) \subset z(\alpha)$, $k = 1, 2, 3, \dots$.

If G is finite one easily obtains a contradiction, whereas if G is infinite then one sees that $t^k \neq e$, $k = 1, 2, 3, \dots$.

Consider then the infinite discrete group H generated by t and the restriction of h to H . Clearly the restriction of h is a nonconstant almost periodic function on H . But if $\{n_k\}$ is any sequence of positive integers such that $n_{k+1} - n_k > 2$, then for $k > 1$ it is quickly verified that

$$\|T_{t^{n_k}} h - T_{t^{n_1}} h\|_{\infty} \geq |h(t)| \neq 0,$$

thus contradicting the almost periodicity of h restricted to H . Thus $\alpha \equiv 0$. $\#$

Obviously the preceding theorem allows us to assume that $\alpha(e) \neq 0$ if $\alpha \neq 0$ is a solution of the functional equation. This reduction permits a considerable strengthening of Lemma 1.

LEMMA 2. Let G be a LCA group and suppose h, α are functions on G such that $h(t) h(s) \alpha(ts) = \alpha(t) \alpha(s)$, $t, s \in G$. If $e \in \tilde{z}(\alpha)$ then:

i) $\alpha(e) = h(e)^2$.

ii) If $t \in \tilde{z}(\alpha)$ then $h(t) = h(e)$.

iii) If $t \in \tilde{z}(\alpha)$ then

$$\alpha(t^k) = \alpha(e) \left(\frac{\alpha(t)}{\alpha(e)} \right)^k,$$

and

$$h(t^k) = h(e), \quad k = 0, 1, 2, \dots$$

iv) If $z(\alpha) = \phi$ then h is a nonzero constant.

An easy consequence of the lemma is the following theorem.

THEOREM 7. Let G be a LCA group, $h \neq 0$ an almost periodic function on G , and suppose that $h(t)h(s)\alpha(ts) = \alpha(t)\alpha(s)$, $t, s \in G$.

If h is nonconstant and $z(h) = \phi$ then $\alpha \equiv 0$.

PROOF. If $\alpha \neq 0$ then by Theorem 6 we have $e \in \hat{Z}(\alpha)$. But then for any $t \in G$ the assumption $z(h) = \phi$ implies via

$$0 \neq h(t)h(t^{-1})\alpha(e) = \alpha(t)\alpha(t^{-1})$$

that $t \in \hat{Z}(\alpha)$, that is, $z(\alpha) = \phi$. The previous lemma then shows that h is a constant, contrary to assumption. $\#$

REMARK. The theorem is actually valid without any assumption of almost periodicity on h .

We note by Theorem 7 that if $\alpha \neq 0$ and h is nonconstant then $z(h) \neq \phi$. In a short while we shall make use of this observation to show that no nonzero solutions exist when G is connected. First we state another important lemma.

LEMMA 3. Let G be a LCA group, $h \neq 0$ an almost periodic function on G and suppose $h(t)h(s)\alpha(ts) = \alpha(t)\alpha(s)$, $t, s \in G$. Then the following are equivalent:

i) $\alpha \neq 0$

ii) $\hat{Z}(\alpha)$ is a closed subgroup of G .

THEOREM 8. Let G be a LCA group, h a nonconstant almost periodic function on G , and suppose that α is a continuous function on G such that $h(t)h(s)\alpha(ts) = \alpha(t)\alpha(s)$, $t, s \in G$. If G is connected then $\alpha \equiv 0$.

PROOF. The previous remark combined with Lemma 1 ii) and Lemma 3 show that if $\alpha \neq 0$ then $\hat{Z}(\alpha)$ is a proper nonempty open and closed subset of G , contradicting the connectedness of G . $\#$

REMARKS. a) Lemma 2 ii) and Lemma 3 combine to show that if $\alpha \neq 0$ then $\hat{Z}(\alpha)$ is a closed subgroup of G which is

contained in $\{t \mid h(t) = h(e)\} \subset \tilde{Z}(h)$.

b) If $\alpha \neq 0$ and G is infinite then one can show that $\tilde{Z}(\alpha) \neq \{e\}$, whereas if G is finite then one may have $\tilde{Z}(\alpha) = \{e\}$. Similarly it is possible that $\tilde{Z}(\alpha)$ can be either a proper or improper subset of $\{t \mid h(t) = h(e)\}$.

We have now reached the penultimate stage in our discussion of the functional equation. We have solutions whenever h is a constant, and know that in other cases nonzero solutions will exist only if $e \in \tilde{Z}(h)$, $\tilde{Z}(h) \neq \emptyset$ and G is disconnected. Straight forward arguments combined with previous remarks, and the following lemma will establish our final theorem stated below.

LEMMA 4. Let G be a LCA group, h an almost periodic function on G , and suppose there exists $\alpha \neq 0$ such that $h(t) h(s) \alpha(ts) = \alpha(t) \alpha(s)$, $t, s \in G$. Then:

- i) If $t, s \in \tilde{Z}(h) \cap \tilde{Z}(\alpha)$ then $ts \in \tilde{Z}(\alpha)$.
- ii) If $t \in \tilde{Z}(h) \cap \tilde{Z}(\alpha)$ then $t^{-1} \in \tilde{Z}(h)$
- iii) If $t \notin \tilde{Z}(\alpha)$ then either $t \notin \tilde{Z}(h)$ or $t^{-1} \notin \tilde{Z}(h)$.
- iv) $\tilde{Z}(\alpha)$ contains every symmetric subset of $\tilde{Z}(h)$.
- v) $\tilde{Z}(\alpha)$ contains every subgroup of G which is contained in $\tilde{Z}(h)$.
- vi) $G / \tilde{Z}(\alpha)$ is finite.

THEOREM 9. Let G be a disconnected LCA group, h a non-constant almost periodic function on G . Then the following are equivalent:

- i) There exists a continuous function $\alpha \neq 0$ such that $h(t) h(s) \alpha(ts) = \alpha(t) \alpha(s)$, $t, s \in G$.
- ii) There exists a unique open and closed subgroup $K \subset \tilde{Z}(h)$ such that:
 - a) $h(t) = h(e)$, $t \in K$.
 - b) If $t, s \in \tilde{Z}(h) \cap K$ then $ts \in K$.

Moreover if $\alpha \neq 0$ is a solution of the functional equation

then $\tilde{Z}(\alpha) = K$ and there exists a unique $\gamma \in \hat{K}$ such that

$$\alpha(t) = \begin{cases} h(e)^2 (t, \gamma), & t \in K \\ 0 & , t \notin K, \end{cases}$$

and every such α is a nonzero continuous solution of the functional equation.

REMARKS. a) The assertions in Lemma 4 are not logically independent. Indeed it is easily seen that $i) \Rightarrow ii) \Leftrightarrow iii) \Leftrightarrow iv) \Rightarrow v)$, and $i) \Rightarrow vi)$. However it is not the case that $ii) \Rightarrow i)$, $v) \Rightarrow iv)$ or $vi) \Rightarrow i)$.

b) It would be interesting to discover more amenable conditions for the existence of the subgroup K , and hence for the existence of the solution α , than those indicated in the theorem.

Before we return to a consideration of the algebras $\mathcal{L}_1(h)$ let us give several examples of functions h for which there may or may not exist solutions to the functional equation.

EXAMPLE 1. Let G be a disconnected LCA group, h a non-constant almost periodic function on G such that

$$h(t) = \begin{cases} h(e), & t \in \tilde{Z}(h) \\ 0 & , t \notin \tilde{Z}(h). \end{cases}$$

If $\tilde{Z}(h)$ is a group then there exists a continuous solution $\alpha \neq 0$ to the functional equation. In this case $K = \tilde{Z}(\alpha) = \tilde{Z}(h)$.

A concrete instance would be where $G = \mathbb{Z}$, the additive group of the integers, and $h(n) = \cos^2 \frac{\pi n}{2}$, $n \in \mathbb{Z}$.

EXAMPLE 2. Let G be a discrete LCA group, h a non-constant positive definite almost periodic function on G such that $h(e) \neq 0$ and the range of h consists of only the values $h(e)$ and zero. Then there exists a continuous solution $\alpha \neq 0$.

Theorem 9 may be applied here since the hypotheses on h insure

that $\tilde{z}(h)$ is a group [2]. Once again $K = \tilde{z}(\alpha) = \tilde{z}(h)$.

EXAMPLE 3. Let $G = \mathbb{Z}$ and set $h(n) = \cos \frac{\pi n}{2}$, $n \in \mathbb{Z}$. Then there exists no solution α of the functional equation except $\alpha \equiv 0$. This follows at once from Lemma 4 iv), since $\tilde{z}(\alpha) \subset \{n \mid h(n) = h(e)\} = \{4k\} \subset \{2k\} = \tilde{z}(h)$ shows that $\tilde{z}(\alpha)$ does not contain every symmetric subset of $\tilde{z}(h)$.

EXAMPLE 4. Let $G = \mathbb{Z}$ and set

$$h(n) = \frac{1}{2i} \left[e^{\frac{i\pi n}{2}} - e^{-\frac{i\pi n}{2}} + 2i \cos^2 \pi n \right], n \in \mathbb{Z}.$$

Using Theorem 9 it is easy to see that no solution $\alpha \neq 0$ exists since the only candidate for K is $\{2k\}$ which does not satisfy condition ii) b of the theorem. More generally one can show that $\tilde{z}(h)$ and $K = \tilde{z}(\alpha)$ satisfy ii) - vi) of Lemma 4 but not i).

REMARK. For finite G it is possible that solutions of the functional equation exist and $\tilde{z}(\alpha) \neq \tilde{z}(h)$. We do not know if this is possible for infinite groups.

5. APPLICATION TO THE MULTIPLICATIVE LINEAR FUNCTIONALS ON $\mathcal{L}_1(h)$. Now let us apply the development of the previous section to describe the multiplicative linear functionals on the algebras $\mathcal{L}_1(h)$. We make the following definition.

DEFINITION 4. A nonassociative Banach algebra A is said to be a radical algebra if the only continuous homomorphism from A to \mathbb{C} is the zero homomorphism.

Combining Theorem 4 with the results of the previous section yields the next theorem.

THEOREM 10. Let G be a LCA group, $h \neq 0$ a trigonometric polynomial on G , $\|h\|_\infty \leq 1$.

1) $\mathcal{L}_1(h)$ is a radical algebra if any one of the following conditions is satisfied:

i) h is nonconstant and G is connected.

ii) h is nonconstant and $z(h) = \emptyset$.

iii) $h(e) = 0$.

iv) G is infinite, $h(e) \neq 0$ and $\hat{Z}(h)$ contains no non-trivial subgroups.

2) If h is a constant then the multiplicative functionals on $\mathcal{L}_1(h)$ are precisely those continuous linear functionals on $\mathcal{L}_1(h)$ of the form

$$F_{\gamma}(f) = \int_G f(t) \alpha_{\gamma}(t) dm(t), \quad f \in \mathcal{L}_1(h),$$

where

$\alpha_{\gamma}(t) = h(e)^2 (t, \gamma)$, $t \in G$ and $\gamma \in \hat{G}$. Moreover the correspondence $\gamma \longleftrightarrow \alpha_{\gamma}$ is bijective.

3) If G is disconnected and h is nonconstant then the following are equivalent:

i) $\mathcal{L}_1(h)$ is not a radical algebra.

ii) There exists a unique open and closed subgroup $K \subset \hat{Z}(h)$ such that:

a) $h(t) = h(e)$, $t \in K$.

b) If $t, s \in \hat{Z}(h) \setminus K$ then $ts \notin K$.

Furthermore, if $\mathcal{L}_1(h)$ is not a radical algebra then the multiplicative linear functionals on $\mathcal{L}_1(h)$ are precisely those continuous linear functionals on $\mathcal{L}_1(h)$ of the form

$$F_{\gamma}(f) = \int_G f(t) \alpha_{\gamma}(t) dm(t), \quad f \in \mathcal{L}_1(h),$$

where

$$\alpha_{\gamma}(t) = \begin{cases} h(e)^2 (t, \gamma), & t \in K \\ 0 & , t \notin K \end{cases}$$

and $\gamma \in \hat{K}$. Moreover the correspondence $\gamma \longleftrightarrow \alpha_{\gamma}$ is bijective.

From this theorem it is apparent that the most interesting situation is where G is disconnected, h nonconstant, and $\mathcal{L}_1(h)$

is not a radical algebra. In the remainder of our discussion we shall, for the most part, restrict our attention to this case. To avoid needless repetition we make the following definition.

DEFINITION 5. If G is a disconnected LCA group, h a non-constant trigonometric polynomial on G , $\|h\|_{\infty} \leq 1$, and $\mathcal{L}_1(h)$ is not a radical algebra then the unique open and closed subgroup K described in Theorem 10-3) ii) will be called the solution group for $\mathcal{L}_1(h)$.

In the situation described in the third portion of Theorem 10 we can easily define a Fourier transform on $\mathcal{L}_1(h)$, namely by setting for each $f \in \mathcal{L}_1(h)$,

$$\begin{aligned} \overset{o}{f}(\gamma) &= \int_G f(t) \alpha_{\gamma^{-1}}(t) \, dm(t) \\ &= h(e)^2 \int_K f(t) (t^{-1}, \gamma) \, dm(t), \quad \gamma \in \hat{K}. \end{aligned}$$

It is easily seen that the mapping $f \rightarrow \overset{o}{f}$ defines a continuous algebra homomorphism of $\mathcal{L}_1(h)$ into $C_0(\hat{K})$, the Banach algebra of continuous functions on \hat{K} which vanish at infinity.

Moreover in this context if $f \in \mathcal{L}_1(h)$ then $f_2 = \chi_K f$ also belongs to $\mathcal{L}_1(h)$, and can also, in an obvious manner, be considered as an element of the group algebra $L_1(K)$. From the definition of $\overset{o}{f}$ we see at once that $\overset{o}{f} = h(e)^2 \hat{f}_2$, where $\hat{}$ denotes the usual Fourier transform in $L_1(K)$.

Moreover if $f \in \mathcal{L}_1(h)$ and we set $f_1 = f - f_2 = f - \chi_K f$, then it is clear that f_1 can be considered both as an element of $\mathcal{L}_1(h)$ and of $L_1(G \sim K)$, where the latter notation now designates only the Banach space of functions integrable with respect to Haar measure restricted to $G \sim K$. $G \sim K$ is of course not in general a group. We shall always use f_1, f_2 to denote the functions defined as above.

These observations show that if $\mathcal{L}_1(h)$ is not a radical algebra then it can be considered as a sum of $L_1(K)$ and $L_1(G \sim K)$. We shall see momentarily that somewhat more can be

asserted, but first we need another definition.

DEFINITION 6. If A is a nonassociative Banach algebra then the radical of A , denoted by Rad , is the subset of A consisting of all elements of A which are mapped into zero by all continuous complex homomorphisms of A , that is, Rad is the intersection of the kernels of the continuous multiplicative linear functionals on A .

REMARK. It is evident that Rad is a closed ideal in A .

The proof of the following decomposition theorem is not difficult.

THEOREM 11. Let G be a disconnected LCA group, h a non-constant trigonometric polynomial on G , $\|h\|_{\infty} \leq 1$, and suppose $\mathcal{L}_1(h)$ is not a radical algebra. If K is the solution group for $\mathcal{L}_1(h)$ then:

- i) $\text{Rad} = \{f \mid f \in \mathcal{L}_1(h), f = 0 \text{ a.e. on } K\}$.
- ii) $L_1(K)$ is an associative subalgebra of $\mathcal{L}_1(h)$.
- iii) $\mathcal{L}_1(h) = \text{Rad} \oplus L_1(K)$.
- iv) There exists a homeomorphic algebra isomorphism of $\mathcal{L}_1^{(h)}/\text{Rad}$ onto $L_1(K)$.

REMARKS. a) Parts iii) and iv) of the theorem show that under the given assumptions the Wedderburn first principal structure theorem holds for $\mathcal{L}_1(h)$ [8, p. 59].

b) Moreover the theorem also provides examples of Banach algebras where the Wedderburn theorem holds but the sufficient conditions utilized by Feldman [3] to insure the validity of this theorem for Banach algebras may not be valid. In particular it is evident that neither Rad nor $\mathcal{L}_1^{(h)}/\text{Rad}$ need be finite dimensional [3, p.776].

c) Also the fact that Rad can be identified with $L_1(G \sim K)$ shows that this latter Banach space is an algebra under the multiplication \circ . This is a direct reflection of the property of the

solution group K that $(\hat{Z}(h) \sim K) (\hat{Z}(h) \sim K) \subset G \sim K$.

6. THE QUESTION OF ASSOCIATIVITY. The decomposition theorem of the preceding section allows us to gain further insight into the question of when $\mathcal{L}_1(h)$ is associative, that is, when is $\mathcal{L}_1(h)$ a Banach algebra in the usual sense of the term. First we state a lemma which will be useful now and with respect to investigations of the ideal structure of $\mathcal{L}_1(h)$.

LEMMA 5. Let G be a disconnected LCA group, h a non-constant trigonometric polynomial on G , $\|h\|_\infty \leq 1$. Suppose $\mathcal{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathcal{L}_1(h)$. Then the following are equivalent:

- i) $K = \hat{Z}(h)$.
- ii) $f \circ g = 0$, $f \in \text{Rad}$, $g \in L_1(K)$.
- iii) $f \circ g = 0$, $f, g \in \text{Rad}$.
- iv) $f \circ g = 0$, $f \in \text{Rad}$, $g \in \mathcal{L}_1(h)$.

We note that when Theorem 11 is valid that if $f, g \in \mathcal{L}_1(h)$ then $f_2 \circ g_2 = h(e)^2 f_2 * g_2$, while if $K = \hat{Z}(h)$ then $f \circ g = h(e)^2 f_2 * g_2$. The latter assertion follows from the lemma. Utilizing these observations and the lemma one easily establishes the next result.

THEOREM 12. Let G be a disconnected LCA group, h a non-constant trigonometric polynomial on G , $\|h\|_\infty \leq 1$. Suppose $\mathcal{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathcal{L}_1(h)$. If $K = \hat{Z}(h)$ then $\mathcal{L}_1(h)$ is a commutative (associative) Banach algebra.

PROOF. If $f, g, k \in \mathcal{L}_1(h)$ then

$$\begin{aligned} (f \circ g) \circ k &= (f_2 \circ g_2) \circ k_2 = h(e)^4 (f_2 * g_2) * k_2 \\ &= h(e)^4 f_2 * (g_2 * k_2) \\ &= f \circ (g \circ k). \quad \# \end{aligned}$$

We do not know if the converse of this theorem is valid, that is, whether $K = \tilde{Z}(h)$ is a necessary and sufficient condition for the associativity of $\mathcal{L}_1(h)$. Some support for such a conjecture is provided by the following theorem.

THEOREM 13. Let G be a discrete LCA group, h a non-constant trigonometric polynomial on G , $\|h\|_\infty \leq 1$. Suppose $\mathcal{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathcal{L}_1(h)$. Then the following are equivalent:

- i) $K = \tilde{Z}(h)$.
- ii) $\mathcal{L}_1(h)$ is a commutative (associative) Banach algebra.

REMARK. A little more can be said namely that $\mathcal{L}_1(h)$ is not associative whenever $\tilde{Z}(h)$ is closed and $\tilde{Z}(h) \cap K \neq \emptyset$.

The preceding two theorems show that it is of some interest to have criteria which will insure that $K = \tilde{Z}(h)$. The best result in this direction is contained in the next theorem. If $f \in \mathcal{L}_1(h)$ we shall set $f^*(t) = \overline{f(t^{-1})}$. As is well known the mapping $f \rightarrow f^*$ is an involution on the group algebra $L_1(G)$, that is $f^{**} = f$, $(\lambda f)^* = \overline{\lambda} f^*$ and $(f * g)^* = f^* * g^*$. We define an involution on $\mathcal{L}_1(h)$ in the obvious fashion.

THEOREM 14. Let G be a disconnected LCA group, $h(t) = \sum_{k=1}^n c_k(t, \chi_k)$ a nonconstant trigonometric polynomial on G , $\|h\|_\infty \leq 1$. Suppose $\mathcal{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathcal{L}_1(h)$. Then the following are equivalent:

- i) The mapping $f \rightarrow f^*$ is an involution on $\mathcal{L}_1(h)$.
- ii) The c_k , $k = 1, 2, \dots, n$, are either all real or all pure imaginary.
- iii) $\tilde{Z}(h) = K$ and $h(e)$ is real or pure imaginary.

REMARK. Actually i) and ii) are equivalent for arbitrary $\mathcal{L}_1(h)$.

7. IDEAL STRUCTURE. In this section we shall present some results on the ideal structure of $\mathcal{L}_1(h)$, particularly in case where $\hat{z}(h) = K$. In order to make the results easily accessible we shall present them in a tabular form. Standard terminology from the theory of Banach algebras will not be defined, the definitions can be carried over verbatim to the algebras $\mathcal{L}_1(h)$. If the reader wishes he can refresh his memory with [7,8]. The proofs of the assertions below are all relatively straight forward and rely heavily on Theorem 11 and Lemma 5.

I. Assume G is a disconnected LCA group, h a nonconstant trigonometric polynomial on G , $\|h\|_\infty \leq 1$, $\mathcal{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathcal{L}_1(h)$. We shall call a linear subspace $I \subset \mathcal{L}_1(h)$ a K-ideal of $\mathcal{L}_1(h)$ whenever $f_2 \circ g \in I$ whenever $f_2 \in L_1(K)$, $g \in I$. I will always denote a linear subspace in $\mathcal{L}_1(h)$.

I is a

closed ideal	\implies	$I = I_1 \oplus I_2$, $I_1 \subset \text{Rad}$ a closed ideal in $\mathcal{L}_1(h)$, $I_2 \subset L_1(K)$ a closed ideal in $L_1(K)$. (I_2 need not be an ideal in $\mathcal{L}_1(h)$, and converse fails.)
closed ideal, and $I_2 \subset L_1(K)$ is a closed ideal in $\mathcal{L}_1(h)$.	\iff	$\text{Rad} \circ I_2 = \{0\}$.
closed K-ideal	\iff	$I = I_1 \oplus I_2$, $I_1 \subset \text{Rad}$ is a closed K-ideal in $\mathcal{L}_1(h)$, $I_2 \subset L_1(K)$ is a closed ideal in $L_1(K)$.
proper regular ideal in $\mathcal{L}_1(h)$ of the form $I = \text{Rad} \oplus I_2$	\longleftarrow	$I_2 \subset L_1(K)$ is a proper regular ideal in $L_1(K)$.

proper regular ideal in $\mathcal{L}_1(h)$	\implies	$I = I_1 \oplus I_2, I_2 \subset L_1(K)$ is a proper regular ideal in $L_1(K)$.
proper maximal regular ideal	\iff	$I = \text{Rad} \oplus I_2, I_2 \subset L_1(K)$ is a proper maximal regular ideal in $L_1(K)$.
closed ideal such that $kh(I) = I$	\iff	$I = \text{Rad} \oplus I_2, I_2 \subset L_1(K)$ is a closed ideal such that $kh(I_2) = I_2$.

Under the assumptions of this section we also have the following two facts.

- i) K is discrete if and only if Rad is a closed regular ideal in $\mathcal{L}_1(h)$.
- ii) $K = \hat{\mathcal{Z}}(h)$ if and only if the collections of ideals and K -ideals in $\mathcal{L}_1(h)$ are identical.

II. Same assumptions as in I plus $\hat{\mathcal{Z}}(h) = K$.

I is a

closed ideal	\iff	$I = I_1 \oplus I_2, I_1 \subset \text{Rad}$ is a closed linear subspace of $\mathcal{L}_1(h), I_2 \subset L_1(K)$ is a closed ideal in $L_1(K)$.
proper regular ideal	\iff	$I = \text{Rad} \oplus I_2, I_2 \subset L_1(K)$ is a proper regular ideal in $L_1(K)$.
primary ideal	\iff	$I = I_1 \oplus I_2, I_1 \subset \text{Rad}$ a closed linear subspace of $\mathcal{L}_1(h), I_2 \subset L_1(K)$ is a maximal regular ideal in $L_1(K)$.
closed ideal	\longleftarrow	I is a closed linear subspace invariant under translation by K . (Converse is false).

Some additional results and observations are as follows.

- i) $\mathcal{L}_1(h)$ contains primary ideals which are not maximal, contrary to the situation in $L_1(G)$.
- ii) Without assuming h is nonconstant one can show that every primary ideal in $\mathcal{L}_1(h)$ is maximal if and only if h is a constant.
- iii) The collection of closed linear subspaces in $\mathcal{L}_1(h)$ invariant under translation by $G(K)$ is not identical with the collection of closed ideals (K -ideals).

Before we can discuss the last set of results in this section we need to introduce some additional notation. We set

$$\mathcal{L}_1^R(h) = \{f \mid f \in \mathcal{L}_1(h), f \text{ real valued}\}, L_1^R(G) =$$

$\{f \mid f \in L_1(G), f \text{ real valued}\}$. It is easily seen that $L_1^R(G)$ is a real Banach algebra under $*$, and that $\mathcal{L}_1^R(h)$ is also such an algebra under \circ provided we assume that $h(e)^2$ is real and maintain the assumptions of II above. We shall denote subsets of $\mathcal{L}_1^R(h)$ ($L_1^R(G)$) by a superscript R . In [1] Aubert has studied the existence and characterization of certain types of convex ideals in $L_1^R(G)$. He shows that I^R is a convex maximal regular ideal in $L_1^R(G)$ if and only if $I^R = \{f \mid f \in L_1^R(G), \hat{f}(e) = 0\}$; and that $L_1^R(G)$ contains no closed or regular absolutely convex

ideals. (The reader is referred to [1] for the definitions of convexity). The situation in $\mathcal{L}_1^R(h)$ is slightly different.

III. Same assumptions as II plus $h(e)^2$ is real.

I^R is a

convex maximal regular ideal in $\mathcal{L}_1^R(h)$	\iff	$I^R = \{f \mid f \in \mathcal{L}_1^R(h), \hat{f}(e) = 0\}$ $= \text{Rad} \oplus \{f \mid f \in L_1^R(K), \hat{f}(e) = 0\}.$
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convex ideal in $\mathcal{L}_1^R(h)$ which is the intersec- tion of maximal regular ideals in $\mathcal{L}_1^R(h)$.	\Longleftrightarrow	$I^R \subset \{f \mid f \in \mathcal{L}_1^R(h), \overset{o}{f}(e) = 0\}.$
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If we also assume that K is discrete then:

I^R is a

proper regular absolutely con- vex ideal in $\mathcal{L}_1^R(h)$.	\Longleftrightarrow	$I^R = \text{Rad}^R.$
closed absolutely convex ideal in $\mathcal{L}_1^R(h)$ which is contained in Rad^R	\Longleftrightarrow	There exists a subset $E \subset G \cup K$ such that $I^R = \{f \mid f \in \mathcal{L}_1^R(h), f = 0 \text{ c.a.e. on } E \cup K\}.$

Two additional facts worth mentioning are:

- i) If we assume besides the assumptions in III that K is nondiscrete then there exist no proper regular absolutely convex ideals in $\mathcal{L}_1^R(h)$.
- ii) In any case, closed absolutely convex ideals always exist, for example Rad^R and $L_1^R(K)$ are two such ideals in $\mathcal{L}_1^R(h)$.

8. THE MULTIPLIERS FOR $\mathcal{L}_1(h)$. Suppose A is a commutative Banach algebra without order, that is, $fg = 0, g \in A$, implies $f = 0$. A multiplier for such a Banach algebra is a mapping $S: A \rightarrow A$ such that $(Sf)g = f(Sg), f, g \in A$. It is well known that the multipliers for commutative Banach algebras without order are precisely the bounded linear operators S on A such that $(Sf)g = f(Sg)$, and that the set of multipliers for A forms a

commutative closed subalgebra of the algebra of all bounded linear operators on A [11]. We can not apply these definitions directly to the algebras $\mathcal{L}_1(h)$, in the first place because they may not be associative, and in the second because they are never without order if h is nonconstant.

Consequently we shall modify the usual definition of a multiplier in such a way that the properties associated with such operators are preserved and so that the definition is applicable to $\mathcal{L}_1(h)$.

DEFINITION 7. Let G be a LCA group, $h \neq 0$ a trigonometric polynomial on G , $\|h\|_\infty \leq 1$. A multiplier S for $\mathcal{L}_1(h)$ is a bounded linear operator on $\mathcal{L}_1(h)$ such that $(Sf) \circ g = f \circ (Sg)$, $f, g \in \mathcal{L}_1(h)$.

Below we shall give a theorem which provides a complete description of the multipliers for $\mathcal{L}_1(h)$ in the case where $\mathcal{L}_1(h)$ is not a radical algebra and $\tilde{Z}(h)$ is equal to the solution group. First, however, in order to make the theorem understandable we must introduce some additional notation.

If B is a Banach space then $E(B)$ will denote the Banach algebra of all bounded operators on B , and if B is itself a commutative Banach algebra without order then $M(B)$ will denote the closed commutative subalgebra of $E(B)$ consisting of the multipliers for B . The linear space of multipliers for $\mathcal{L}_1(h)$ will be denoted by $M[\mathcal{L}_1(h)]$. Clearly it is a closed linear subspace of $E[\mathcal{L}_1(h)]$. When $\mathcal{L}_1(h) = \text{Rad} \oplus L_1(K)$ then P_2 shall denote the projection of $\mathcal{L}_1(h)$ onto $L_1(K)$.

We can now state the indicated theorem

THEOREM 15. Let G be a disconnected LCA group, h a non-constant trigonometric polynomial on G , $\|h\|_\infty \leq 1$. Suppose $\mathcal{L}_1(h)$ is not a radical algebra, K is the solution group for $\mathcal{L}_1(h)$ and $K = \tilde{Z}(h)$. Then the following are equivalent:

- i) $S \in M[\mathcal{L}_1(h)]$.

- ii) There exists a unique $S_1 \in E[L_1(G \sim K)]$,
 $S_2 \in M[L_1(K)]$ such that $Sf = S_1 f_1 + S_2 f_2$ for each
 $f = f_1 + f_2$ in $\mathcal{L}_1(h) = \text{Rad} \oplus L_1(K)$.
- iii) $S \in E[\mathcal{L}_1(h)]$ is such that
- a) $S: \text{Rad} \rightarrow \text{Rad}, S: L_1(K) \rightarrow L_1(K)$.
 - b) $S P_2 T_s = T_s S P_2, s \in K$.

Moreover $M[\mathcal{L}_1(h)]$ is a closed subalgebra of $E[\mathcal{L}_1(h)]$ and the correspondence determined by the relationship in ii) defines an isometric algebra isomorphism of $M[\mathcal{L}_1(h)]$ onto $E[L_1(G \sim K)] \oplus M[L_1(K)]$, where $M[L_1(K)]$ is algebraically isometrically isomorphic with $\{S \mid S \in M[\mathcal{L}_1(h)], Sf_1 = 0, f_1 \in \text{Rad}\}$.

REMARKS. a) Clearly $M[\mathcal{L}_1(h)]$ is not commutative.

b) As is well known [9] the multipliers for the group algebra $L_1(G)$ can be identified with all the bounded linear operators on $L_1(G)$ which commute with translation (and also with the bounded regular Borel measures on G). This is no longer the case for $\mathcal{L}_1(h)$. Indeed one can even show that if $s \in G \sim K$ then T_s is not a multiplier for $\mathcal{L}_1(h)$. The best that can be said is contained in iii) above.

c) Of course if h is a constant then the problem reduces essentially to studying the multipliers for $L_1(G)$.

d) We know nothing about the multipliers when $\mathcal{L}_1(h)$ is a radical algebra.

9. $\mathcal{L}_1(h)$ FOR ALMOST PERIODIC h . With the exception of Theorem 14 all of the results in the previous sections remain valid if the trigonometric polynomial h is replaced by an almost periodic function. If one does this however, then the motivation given in the first section for investigating the algebras $\mathcal{L}_1(h)$ is no longer directly applicable since we do not know of any characterization of measures of the form $du = h dm$, where h is an almost

periodic function, in terms of intrinsic properties of the measure μ which are generalizations of well known properties of Haar measure m . Such a connection undoubtedly exists and it would be of some interest to uncover it.

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